

# QCD improved exclusive rare $B$ -Decays at the heavy b-quark limit

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## ABSTRACT

The renormalization effects from the b-quark scale down to the non-perturbative QCD regime are studied for rare  $B$ -decays at the heavy b-quark limit. Phenomenological consequences of these effects are investigated. We find that the anomalous scaling behavior plays a positive role in making non-perturbative model calculations consistent with recent CLEO measurements of  $B \rightarrow K^* \gamma$ .

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# 1 Introduction

Rare  $B$ -decays are vital testing grounds for the Standard Model and therefore have received a lot of theoretical attention [2, 3, 17]. Besides examining the electroweak theory at the one-loop level, and providing quantitative information on the yet undetermined top quark mass and CKM matrix elements  $V_{td}, V_{ts}, V_{tb}$ , radiative rare  $B$ -decays could also be sensitive to new physics beyond the Standard Model. Recently, the CLEO group has reported the first observation of exclusive decay mode,  $B \rightarrow K^*\gamma$ , and the measured average branching ratio,  $BR(B \rightarrow K^*\gamma) = (4.5 \pm 1.5 \pm 0.9) \times 10^{-5}$ , as well as the inclusive upper limit  $BR(B \rightarrow X_s\gamma) < 5.4 \times 10^{-4}$ , are consistent with the Standard Model [1].

To present theoretical estimates for exclusive processes, hadronic matrix elements governed by nonperturbative physics have to be evaluated. A number of effective approaches on the basis of symmetry considerations and phenomenological models for QCD in the nonperturbative regime have been developed and applied to calculations of matrix elements for rare  $B$ -decays [4, 5, 6, 7, 8, 9, 10, 11]. As is well known, certain of these methods, for example, the nonrelativistic quark model, cannot take into account correctly the strong interaction effects at scales much larger than the QCD scale  $\Lambda_{QCD}$ . For current operators containing the heavy quark, the loop corrections with gluonic momenta between  $\Lambda_{QCD}$  and  $m_Q$  lead to large logarithms of the type  $\alpha_s \ln \frac{m_Q}{\Lambda_{QCD}}$ . With renormalization group techniques, these large logarithms can be summed to all orders, resulting in an anomalous scaling factor. This kind of scaling factor has been worked out for the cases such as heavy meson decay constants  $f_M$ ,  $B^0 - \bar{B}^0$  mixing[12], heavy meson semileptonic decay form factors[13], and hadronic  $B$ -decays [14]. In this letter, we will consider the anomalous scaling behavior from the b-quark scale down to the non-perturbative regime for current operators appearing in rare  $B$ -decays upto next-to-leading order at the heavy b-quark limit and present the consequences of this scaling for exclusive rare  $B$ -decays.

In sect. II, we briefly review the renormalization from  $M_W$  to  $m_b$  for rare decay operators in the full QCD with five quarks . More or less, we follow a similar procedure in discussing the anomalous scale below  $m_b$ , which is presented in sect. III. The consequences of this QCD scaling for exclusive rare radiative  $B$ -decays are shown in sect. IV. We summarize in sect. V.

## 2 Renormalization in the full QCD

To illustrate the general procedure of analyzing the anomalous scaling behavior through the renormalization group techniques, we briefly review the renormalization of current operators from heavy particle scale, such as the top quark and W boson, down to the bottom quark scale in QCD with five flavors. At the moment, we ignore the running of

the strong coupling constant between  $m_t$  and  $M_W$ . For the sake of concentration, we consider the rare radiative bottom quark decay. With no strong interaction in the form of QCD, the effective hamiltonian for the rare radiative bottom quark decay, after the t-quark and W-boson are integrated out from the (1-loop) photonic penguin diagrams, reads

$$\mathcal{H}_{eff} = -\frac{2G_F}{\sqrt{2}}(s_3 + s_2 e^{i\delta})A(x_t)\left(\frac{\alpha m_b}{4\pi}\right)\bar{s}_L\sigma^{\mu\nu}b_RF_{\mu\nu}, \quad (1)$$

with  $x_t = m_t^2/M_W^2$ . Here the contributions from virtual up and charm quarks have been ignored and a term proportional to the strange quark mass is dropped considering that  $m_b \gg m_s$ .  $A(x)$  is the Inami-Lim function with the form

$$A(x) = x \left[ \frac{\frac{2}{3}x^2 + \frac{5}{12}x - \frac{7}{12}}{(x-1)^3} - \frac{(\frac{3}{2}x^2 - x) \ln x}{(x-1)^4} \right]. \quad (2)$$

The QCD effects play an important role in enhancing the radiative rare  $B$ -decay, where the usual GIM suppression factor in FCNC processes,  $\frac{m_q}{m_t}$ , is replaced by a logarithmic one  $\ln(\frac{m_t}{m_b})$ . Switching on strong interactions, the effective hamiltonian at  $\mu \ll M_W$  is determined by a proper operator basis. Following the notation of ref. [3], we have

$$\mathcal{H}_{eff} = \frac{4G_F}{\sqrt{2}}(s_3 + s_2 e^{i\delta})\sum_{j=1}^8 C_j(\mu)O_j(\mu), \quad (3)$$

where

$$\begin{aligned} O_1 &= (\bar{s}_{L\alpha}\gamma^\mu b_{L\alpha})(\bar{c}_{L\beta}\gamma_\mu c_{L\beta}), & O_2 &= (\bar{s}_{L\alpha}\gamma^\mu b_{L\alpha})(\bar{c}_{L\beta}\gamma_\mu c_{L\beta}), \\ O_3 &= (\bar{s}_{L\alpha}\gamma^\mu b_{L\alpha})\sum_q(\bar{q}_{L\beta}\gamma_\mu q_{L\beta}), & O_4 &= (\bar{s}_{L\alpha}\gamma^\mu b_{L\beta})\sum_q(\bar{q}_{L\beta}\gamma_\mu q_{L\alpha}), \\ O_5 &= (\bar{s}_{L\alpha}\gamma^\mu b_{L\alpha})\sum_q(\bar{q}_{R\beta}\gamma_\mu q_{R\beta}), & O_6 &= (\bar{s}_{L\alpha}\gamma^\mu b_{L\beta})\sum_q(\bar{q}_{R\beta}\gamma_\mu q_{R\alpha}), \\ O_7 &= \left(\frac{\alpha m_b}{4\pi}\right)\bar{s}_{L\alpha}\sigma^{\mu\nu}b_{R\alpha}F_{\mu\nu}, & O_8 &= \left(\frac{\alpha m_b}{4\pi}\right)\bar{s}_{L\alpha}\sigma^{\mu\nu}T_{\alpha\beta}^ab_{R\beta}G_{\mu\nu}^a. \end{aligned} \quad (4)$$

with  $\alpha$  and  $\beta$  the color indices.

As the effective hamiltonian is renormalization scale independent, changes of the coefficients with the scale should be compensated by changes of these operators. This leads us to a renormalization group equation

$$\mu \frac{d}{d\mu}C_j(\mu) - \sum_{i=1}^8 \gamma_{ij}(g)C_i(\mu) = 0, \quad (5)$$

where  $\gamma$  is the anomalous dimension matrix of the operator basis  $\{O_j\}$ . Introducing the QCD  $\beta$ -function in terms of

$$\mu \frac{d}{d\mu} = \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g}, \quad (6)$$

the general solution of the above equation can be written in a matrix form as

$$\mathbf{C}(M_W/\mu, g(\mu)) = \exp\left[\int_{g(M_W)}^{g(\mu)} dg \frac{\gamma^T(g)}{\beta(g)}\right] \mathbf{C}(1, g(M_W)). \quad (7)$$

Explicit solutions can be obtained perturbatively. In the leading logarithmic approximation in which the terms like  $(\alpha_s \ln M_W/\mu)^n$  are summed to all orders, the  $\beta$ -function and anomalous dimensions calculated at the one-loop level of QCD are needed, while matching conditions are of the zeroth order. For the four-quark operators  $O_1 - O_6$ , the matching conditions determined by the tree-level W-boson exchange are

$$C_j(M_W) = 0 \quad (j = 1, 3, 4, 5, 6), \quad (8)$$

$$C_2(M_W) = 1. \quad (9)$$

The values of  $C_7(M_W)$  and  $C_8(M_W)$  follow from (one-loop) penguin diagrams. They are

$$C_7(M_W) = -\frac{1}{2}A(x_t), \quad (10)$$

and

$$C_8(M_W) = -\frac{1}{2}D(x_t), \quad (11)$$

with

$$D(x) = \frac{x}{2} \left[ \frac{\frac{1}{2}x^2 - \frac{5}{2}x - 1}{(x-1)^3} - \frac{3x \ln x}{(x-1)^4} \right]. \quad (12)$$

Furthermore at the next-to-leading order, the  $\beta$ -function and anomalous dimensions are calculated to two loops of QCD and matching conditions are of the first order. In this case, the logarithmic terms like  $\alpha_s(\alpha_s \ln M_W/\mu)^n$  are summed to all orders.

For the two-body decay  $\bar{B} \rightarrow X_s \gamma$ , which is modelled by  $b \rightarrow s \gamma$  at the quark level, only the photon magnetic moment type operator  $O_7$  contributes and the QCD corrected inclusive rare radiative decay rate is as follows

$$\Gamma(\bar{B} \rightarrow X_s \gamma) = \frac{G_F^2 m_b^5 \alpha}{32\pi^4} (s_2^2 + s_3^2 + 2s_2 s_3 c_\delta) |C_7(\mu)|^2 \quad (13)$$

Practically, this rate may be normalized to the semileptonic  $B$ -decay. In this way we remove the quark mixing angles in the small mixing limit and reduce the dependence on the bottom quark mass. So we have

$$\Gamma(\bar{B} \rightarrow X_s \gamma) = \frac{6\alpha}{\pi} \frac{|C_7(\mu)|^2}{f(m_c/m_b)} \left(1 - \frac{2\alpha_s(m_b)}{3\pi} g(m_c/m_b)\right)^{-1} \Gamma(\bar{B} \rightarrow X_c l \bar{\nu}_l), \quad (14)$$

where  $f(m_c/m_b) = 0.45$  and  $g(m_c/m_b) = 2.4$  correspond to the phase space factor and the one-loop QCD corrections to the semileptonic decay, respectively.

In the leading logarithmic approximation and with an anomalous dimension matrix for the truncated basis  $O_1$ ,  $O_2$ , and  $O_7$

$$\gamma = \frac{g^2}{16\pi^2} \begin{vmatrix} -1 & 3 & 0 \\ 3 & -1 & 232/81 \\ 0 & 0 & 16/3 \end{vmatrix}, \quad (15)$$

the renormalization coefficient for the operator  $O_7$  has the following analytic form[3]

$$C_7(\mu) = -\eta^{16/23} \left\{ \frac{1}{2} A(x_t) + [(58/135)(\eta^{-10/23} - 1) + (29/189)(\eta^{-28/23} - 1)] \right\}, \quad (16)$$

with  $\eta = \alpha_s(M_W)/\alpha_s(\mu)$ . Ref. [3] has presented a discussion on the validity of the extra approximation with the truncated anomalous dimension matrix and estimated that the error in this coefficient function is less than 15%. Furthermore the calculation of ref. [15] in the leading logarithmic approximation without the truncation of the anomalous dimesion matrix also does not manifest significant change in the above estimates. Recently, the next-to-leading logarithmic effects which weaken the QCD-corrections to the  $b \rightarrow s\gamma$ , have been included partly in ref. [16] and result in a change of 10-20% in the inclusive decay rate.

We notice that the upper bound of theoretical estimate for the inclusive rate is relevant to the renormalization scale which is usually chosen about the b-quark mass. Actually, based on eq. (14) and (16), a renormalization scale extrapolated almost to  $2.0 GeV$  is used to set the upper bound of the estimate[17]

$$BR(\bar{B} \rightarrow X_s + \gamma) = (2 \sim 5) \times 10^{-4}. \quad (17)$$

in the minimal standard model.

### 3 Anomalous scaling at the heavy b-quark limit

In this section, we present an analysis for renormalization of operators considered in the previous section below the bottom quark mass. There are three flavors of quarks in the area concerned. The masses of bottom and strange quarks lie somehow on the upper and lower bounds, respectively, whilst the charm quark locates in the middle. Based on this observation, we will scale down by treating the bottom quark as a heavy particle and integrating it in using an effective theory with four flavors. However, when we proceed into the regime between  $m_c$  and  $m_s$ , where both bottom and charm quarks are heavy, the light flavor number becomes three.

Working in an effective theory where the bottom quark is treated as a heavy particle, we have the following expansion for  $O_j$

$$O_j \cong \sum_l D_{jl}(m_b/\mu, g(\mu)) O'_l(\mu) + \mathcal{O}\left(\frac{1}{m_b}\right), \quad (18)$$

where  $\{O'_l\}$  is an operator basis of the lowest possible dimension. Contributions of higher dimension operators vanish at the heavy bottom quark limit. Operators  $O'_l$  only depend upon the light scale  $\mu$  and the large logarithm  $\ln \frac{m_b}{\mu}$  is transferred into the coefficients  $D_{jl}$ , which obey the following renormalization group equations

$$\left[ \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right] D_{jl}(\mu) - \left[ \sum_k \gamma'_{kl}(g') D_{jk}(\mu) - \sum_i \gamma_{ji}(g) D_{il}(\mu) \right] = 0, \quad (19)$$

where apart from different light flavor counting, the QCD  $\beta$ -function retains the same form as the scale goes down, namely

$$\beta(g) = -g \left[ \beta_0 \frac{g^2}{16\pi^2} + \beta_1 \left( \frac{g^2}{16\pi^2} \right)^2 + \dots \right], \quad (20)$$

with the one-loop (two-loop) coefficient listed in Table 1. Along with  $\gamma'$ , the anomalous dimensions of  $\{O'_l\}$  in the heavy quark effective field theory(HqEFT), there is also a term in the above equation associated with  $\gamma$ , the anomalous dimensions of  $O_j$  in the higher scale theory. As shown explicitly in the appendix, the combination of the coefficients  $C_j$  in eq. (7) with the part of  $D_{jl}$  associated to the anomalous dimensions  $\gamma_{ji}$  gives one the anomalous scaling from  $M_W$  down to  $m_b$ . We will drop the  $\gamma$ -term in eq. (19) when we work below the bottom quark mass.

Since the photon magnetic moment type operator  $O_7$  dominates the rare radiative decay  $b \rightarrow s \gamma$ , we concentrate on it at the present stage. Let us consider the proper operators of dimension three<sup>1</sup> in the effective theory where, to remove the heavy mass,

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<sup>1</sup> At the moment we omit the photon field in  $O_7$  and the bottom quark mass that serves as the matching scale and is going to be set by the solution of  $\bar{m}(m_b) = m_b$ , with  $\bar{m}(\mu)$  the running mass.

the bottom quark field is redefined by

$$h_v(x) = e^{im_b v \cdot x y'} \psi_b(x). \quad (21)$$

Here  $v_\mu$  is the four-velocity of the heavy bottom quark, which is a conserved quantity with respect to the low-energy QCD. Obviously, the first operator should be  $O'_1 = \bar{s}_L \sigma_{\mu\nu} h_{vR}$ . Constructing from the Dirac matrix  $\gamma_\mu$  and the bottom quark velocity which is an external parameter also gives us the antisymmetric current operator  $O'_2 = \bar{s}_L (\gamma_\mu v_\nu - v_\mu \gamma_\nu) h_{vR}$ . Because the large logarithmic term arises only from the vertex of the heavy quark with the gluon, which becomes  $ig\lambda^a v^\mu$  in the heavy quark limit, the renormalization is independent of the spin structure of the inserted current operators and does not mix them up. In this case, the anomalous dimension matrix reduces to a constant one and the renormalization group equation for the coefficients has a quite simple form

$$\left[ \mu \frac{\partial}{\partial \mu} + \beta(g') \frac{\partial}{\partial g'} - \gamma'(g') \right] D_{7l}(\mu) = 0. \quad (22)$$

Expressing the solution of this equation in the same manner as eq. (7) gives

$$D_{7l} = \exp \left[ \int_{g'(m_b)}^{g'(m_c)} dg' \frac{\gamma'(g')}{\beta'(g')} + \int_{g''(m_c)}^{g''(\mu)} dg'' \frac{\gamma''(g'')}{\beta''(g'')} \right] D_{7l}(m_b). \quad (23)$$

The second part of this solution arises when we cross the charm quark mass below which the light flavors reduce to three. The charm quark is only involved as a virtual heavy particle in loops and except different QCD couplings and light flavor numbers being used,  $\gamma''$  is the same as  $\gamma'$  which has the form

$$\gamma' = \gamma'_0 \frac{g'^2}{16\pi^2} + \gamma'_1 \left( \frac{g'^2}{16\pi^2} \right)^2 + \dots, \quad (24)$$

with the so-called “hybrid” anomalous dimension of ref. [12] and the two-loop coefficient [20] shown in Table 1.

As a first step we work in the leading logarithmic approximation and get the solutions

$$D_{71}(\mu) = \left[ \frac{\alpha'_s(m_b)}{\alpha'_s(m_c)} \right]^{-\frac{6}{25}} \left[ \frac{\alpha''_s(m_c)}{\alpha''_s(\mu)} \right]^{-\frac{2}{9}}, \quad (25)$$

and

$$D_{7l}(\mu) = 0, \quad \text{for } l \neq 1. \quad (26)$$

Here the matching conditions  $D_{71}(m_b) = 1$ , and  $D_{7l}(m_b) = 0$  for  $l \neq 1$  have been used. As matrix elements are concerned, the factor containing  $\mu$  should be cancelled by the

$\mu$ -dependence of hadronic states which are relevant to nonperturbative effects. This leaves us the anomalous heavy quark mass dependence

$$\Omega_{LLA}^T = \left[ \frac{\alpha'_s(m_b)}{\alpha'_s(m_c)} \right]^{-\frac{6}{25}} [\alpha''_s(m_c)]^{-\frac{2}{9}}. \quad (27)$$

Then we consider next-to-leading logarithmic contributions. Using the matching condition for the current  $\bar{s}\Gamma b$  to the  $g^2$  order[12, 13]

$$\Gamma \longrightarrow [\Gamma - \frac{g^2(m_b)}{24\pi^2} \gamma^\lambda \not{p}_\Gamma \not{p}_\gamma^\lambda], \quad (28)$$

we find that at  $\mu = m_b$

$$\sigma_{\mu\nu}(\gamma_5) \longrightarrow \sigma_{\mu\nu}(\gamma_5), \quad (29)$$

just as at the zeroth order up to a vertex renormalization factor. Thus we also have only one non-zero coefficient, i.e.  $D_{71}$  at the next-to-leading order. Once again we expand this coefficient function at  $m_b$  in the QCD coupling

$$D_{71}(m_b) = 1 + d_1 \frac{g'^2}{16\pi^2} + \dots, \quad (30)$$

and find the next-to-leading correction to the heavy mass dependence

$$(d_1 + \kappa') \frac{\alpha'_s(m_b)}{4\pi} - \kappa' \frac{\alpha'_s(m_c)}{4\pi} + \kappa'' \frac{\alpha''_s(m_c)}{4\pi}. \quad (31)$$

Here a  $\mu$ -dependence factor that is absorbed into the matrix element is understood. It is worthwhile to point out that the combination  $d_1 + \kappa'$  is renormalization-scheme independent. The  $\kappa$ -parameters determined by

$$\kappa = \frac{\gamma_0}{2\beta_0} \left( \frac{\gamma_1}{\gamma_0} - \frac{\beta_1}{\beta_0} \right), \quad (32)$$

are listed in Table 2. Matching the vertex renormalization to the one loop in the full theory [18] with that in the effective theory [13, 19]

$$(Z_{full} - 1) = (Z_{eff} - 1) + \frac{g'^2}{16\pi^2} d_1 \quad (33)$$

at  $\mu = m_b$  in the  $\overline{MS}$  scheme gives us  $d_1 = -6c_F$ , with  $c_F = 4/3$ .

Finally let us combine eq.(31) with eq. (27) and present the anomalous heavy quark mass dependence for the photon magnetic moment type operator  $O_7$  with both of the leading and the next-to-leading logarithmic contributions

$$\Omega^T = \left[ \frac{\alpha'_s(m_b)}{\alpha'_s(m_c)} \right]^{-\frac{6}{25}} [\alpha''_s(m_c)]^{-\frac{2}{9}} [1 - 0.710\alpha'_s(m_b) + 0.073\alpha'_s(m_c) - 0.060\alpha''_s(m_c)]. \quad (34)$$

The QCD fine structure constant accurate to the second order is

$$\alpha_s(m^2) = \frac{4\pi}{\beta_0 \ln(m^2/\Lambda_{\overline{MS}}^2)} \left[ 1 - \frac{\beta_1 \ln \ln(m^2/\Lambda_{\overline{MS}}^2)}{\beta_0^2 \ln(m^2/\Lambda_{\overline{MS}}^2)} \right]. \quad (35)$$

As the heavy quark mass is concerned, we use the running mass  $\bar{m}(\mu)$  in the  $\overline{MS}$  scheme and take  $m_Q$  as the solution of  $\bar{m}(m_Q) = m_Q$ . In terms of the invariant mass  $\hat{m}$  we have the following form [21]

$$\bar{m}(\mu) = \hat{m} [\ln(\mu/\Lambda_{\overline{MS}})]^{-\frac{\gamma_{m0}}{2\beta_0}} \left[ 1 - \frac{\gamma_{m0}}{2\beta_0} \frac{\beta_1 \ln \ln(\mu^2/\Lambda_{\overline{MS}}^2)}{\beta_0^2 \ln(\mu^2/\Lambda_{\overline{MS}}^2)} + \frac{\kappa_m}{\beta_0 \ln(\mu^2/\Lambda_{\overline{MS}}^2)} \right], \quad (36)$$

where  $\gamma_{m0}$  ( $\gamma_{m1}$ ) is the one-loop (two-loop) coefficient of the mass anomalous dimension (see Table 1). The values of  $\kappa_m$  in terms of eq. (32) are given in Table 2. With invariant masses in ref. [21] and  $\Lambda_{\overline{MS}}$  equal to 0.25 GeV, we obtain scale masses in the  $\overline{MS}$  scheme,  $m_b = 4.39$  GeV and  $m_c = 1.32$  GeV. Fine structure constants in  $\Omega^T$  are

$$\alpha'_s(m_b) = 0.204, \quad \alpha'_s(m_c) = 0.332, \quad \alpha''_s(m_c) = 0.300, \quad (37)$$

which give the following value

$$\Omega^T \cong 1.12 \times 1.31 \times (1 - 0.139) \cong 1.26 \quad (38)$$

Several comments are in order:

- ★ The heavy quark mass dependence in the leading logarithmic approximation is universal to the current operators  $\bar{s}\Gamma b$  with  $\Gamma$  any matrix in the Dirac space. However, the  $\Gamma$ -dependence in the full theory enters into the effective theory at higher orders through matching conditions.

- ★ Next-to-leading corrections are twofold. One comes from two-loop anomalous dimensions and  $\beta$ -function and does not depend on  $\Gamma$ . Another arises from matching conditions at the one-loop level and impacts on the correction differently for different  $\Gamma$ . For instance, the counterpart of  $d_1$  in eq. (30) for vector and axial current operators is  $-3c_F$  [20, 19]. The large value of  $d_1$  for the present case leads to the result that the next-to-leading corrections are dominated by the  $\alpha'_s(m_b)$  term in eq. (34) that weakenes the leading effects.

- ★ As we know, the photon magnetic moment type operator does not mix into four quark operators and the gluon magnetic moment type operator. The QCD corrections for the case in hand still remain unchanged when the mixings of these operators with  $O_7$  are taken into account. But these mixings add extra contributions, which have not been touched in this work and are believed not to change the essential feature of the anomalous mass factor.

## 4 Phenomenological consequence

When a non-perturbation method at the QCD scale is used to evaluate hadronic matrix elements, the heavy-quark-mass dependence arising from renormalization should be taken into account. A number of such methods, such as the constituent quark model (CQM), MIT Bag model, and the heavy quark limit (HQL), have been employed in calculating rare  $B$ -decay matrix elements [6, 8, 9, 10, 11]. As a phenomenological application of our result in the previous section, we modify these calculation by incorporating the anomalous scale below the bottom quark mass. Our improved estimates for the exclusive rare radiative  $B$ -decays are listed in Table 3. The  $R_0$  and  $R'$  are the ratio of exclusive to inclusive decay widths, i.e.

$$R = \frac{\Gamma(\bar{B} \rightarrow K^*\gamma)}{\Gamma(\bar{B} \rightarrow X_s\gamma)}, \quad (39)$$

before and after the modification, respectively. The top quark mass dependence and the coefficient  $C_7$  are removed in this ratio. In the pure CQM and MIT Bag model calculations, the values of  $R_0$  are about 0.05. This yields an exclusive branching ratio of  $Br(\bar{B} \rightarrow K^*\gamma) = 2.1 \times 10^{-5}$ , for a top quark mass of 160 GeV. It is small compared to the mean value of recent CLEO measurements. However, the anomalous scale factor of LLA increases this ratio almost up to 0.11, but the next-to-leading correction weakens this enhancement and results in a value of 0.082 for the improved ratio, which gives a branching ratio of  $3.9 \times 10^{-5}$ . We find that incorporating the anomalous scale mass factor plays a positive role in making CQM and bag model calculations consistent with the CLEO date. Using the range of  $3.5 \sim 12.2$  for  $R$  in ref.[10] obtained through varying the oscillator strength  $\beta$  between  $\beta_K = 0.34$  GeV and  $\beta_B = 0.41$  GeV, we get a range of  $(1.7 \sim 5.7) \times 10^{-5}$  for the branching ratio of  $\bar{B} \rightarrow K^*\gamma$  decays. Our numerical results are slightly different from that of ref.[10] because of a different top quark mass being used. This range is lifted up to  $(2.7 \sim 9.4) \times 10^{-5}$  by the anomalous mass factor. The improved estimates are completely in agreement with preliminary CLEO observations.

Even though the strange quark is not particularly heavy compared to the QCD scale, HQL also presents reasonable estimate for the rare radiative  $B$ -decay. We hope improvements will be included along this approach in the future. As the authors of ref. [9] point out, the large  $R$  at the third row of Table 3 is due to the use of a nonrelativistic recoil momentum. From the theoretical point of view, this is questionable considering that the recoil in the  $B$  to  $K^*$  decay is very large. Experimentally, estimates with this large  $R$  for rare radiative  $B$ -decays are not favored.

## 5 Summary

In this letter, we have made an analysis of the anomalous heavy quark mass dependence of the rare  $B$ -decays by considering the anomalous scale of current operators below the bottom quark mass. Combining this anomalous mass factor evaluated upto next-to-leading order with calculations using non-perturbative models, such as CQM and MIT Bag Model, gives us exclusive rare radiative  $B$ -decay rates which are in excellent agreement with the recent CLEO measurements. On the other hand we observe that HQL also works as a preliminary approximation to  $B \rightarrow K$  processes.

When we work below the bottom quark mass in this letter, mixings of four quark operators with the photon magnetic moment type operator have not been touched. We expect that they do not change the essential feature of the anomalous mass factor. The investigation of these mixings and their phenomenological consequences is under way and will be presented elsewhere.

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## Appendix A

In this appendix, we present the formal solution of the renormalization group equations, eq. (5) for  $\mu < M$  and eq. (19) for  $\mu < m$  in the main text. We will show that the renormalization below  $m$  is effectively irrelevant to the anomalous dimension matrix of operators  $O_j$ .

Once we write down the effective hamiltonian in terms of an operator basis

$$\mathcal{H}_{eff} = \sum_j C_j O_j \quad (A.1)$$

at  $\mu \ll M$ , the coefficient ought to satisfy the renormalization group equation, in a matrix form,

$$(\mu \frac{d}{d\mu} - \gamma^T) \mathbf{C} = 0, \quad (A.2)$$

where  $\gamma$  is the anomalous dimension matrix. Introducing a transformation matrix  $V$  in such a way that

$$(V^{-1} \gamma^T V)_{ij} = \tilde{\gamma}_j \delta_{ij}, \quad (A.3)$$

where  $\tilde{\gamma}_j$  are eigenvalues of the transpose of  $\gamma$ , we may construct an alternate operator basis though the linear combination  $\tilde{\mathbf{O}} = V^T \mathbf{O}$ . The effective hamiltonian can be rewritten in the same structure

$$\mathcal{H}_{eff} = \sum_j \tilde{C}_j \tilde{O}_j. \quad (A.4)$$

The advantage of this basis is that operators are multiplicatively renormalized and coefficients have solutions like

$$\tilde{C}_j = \exp \left[ \int_{g(M)}^{g(\mu)} dg \frac{\tilde{\gamma}_j}{\beta} \right] \tilde{C}_j(M). \quad (A.5)$$

In the region  $\mu \leq m \leq M$  where the proper operator basis is  $\{ O'_l \}$ , there is the operator expansion

$$O_j = \sum_l D_{jl} O'_l, \quad (A.6)$$

and the corresponding renormalization group equation has the form

$$(\mu \frac{d}{d\mu} + \gamma') \mathbf{D}_l - \sum_k \gamma'_{kl} \mathbf{D}_k = 0, \quad (A.7)$$

with  $\gamma'$  the anomalous dimension matrix of  $O'_l$ . Using the transform (A.3) and similarly

$$(W^{-1} \gamma'^T W)_{kl} = \tilde{\gamma}'_l \delta_{kl}, \quad (A.8)$$

with  $\tilde{\gamma}'_l$  the eigenvalues of the transpose of  $\gamma'$ , the coefficients have the following solutions

$$\tilde{D}_{jl} = \exp \left[ - \int_{g(m)}^{g(\mu)} dg \frac{\tilde{\gamma}_j}{\beta} + \int_{g'(m)}^{g'(\mu)} dg' \frac{\tilde{\gamma}'_l}{\beta'} \right] \tilde{D}_{jl}(m). \quad (A.9)$$

In terms of these coefficients, the effective hamiltonian becomes

$$\mathcal{H}_{eff} = \sum_{jl} \tilde{C}_j \tilde{D}_{jl} \tilde{O}'_l = \sum_{jl} \left\{ \exp \left[ \int_{g(M)}^{g(m)} dg \frac{\tilde{\gamma}_j}{\beta} \right] \tilde{C}_j(M) \exp \left[ \int_{g'(m)}^{g'(\mu)} dg' \frac{\tilde{\gamma}'_l}{\beta'} \right] \tilde{D}_{jl}(m) \right\} \tilde{O}'_l. \quad (A.10)$$

It is remarkable that the combination of the first piece of  $\tilde{D}_{jl}$ , which is associated with the anomalous dimensions in the effective theory above  $m$ , with  $\tilde{C}_j$  in eq (A.5) cancels the  $\mu$ -dependence and gives the anomalous scaling behavior from  $M$  down to  $m$ , which is represented by the first coefficient of the effective hamiltonian in eq. (A.10). The second coefficient, representing the anomalous scale from  $m$  down to  $\mu$ , is determined by the anomalous dimensions in the effective theory at  $\mu \leq m$ .

Table 1. Perturbation coefficients for the  $\beta$ -function and anomalous dimensions

$\beta$ -function		anomalous dimensions of the quark mass		anomalous dimensions of $\bar{s}\Gamma b$ in the HqEFT	
$\beta_0$	$\beta_1$	$\gamma_{m0}$	$\gamma_{m1}$	$\gamma'_0$	$\gamma'_1$
$11 - \frac{2}{3}N_f$	$102 - \frac{38}{3}N_f$	8	$\frac{8}{3}(\frac{101}{2} - \frac{5}{3}N_f)$	- 4	$-(\frac{254}{9} + \frac{56}{27}\pi^2 - \frac{20}{9}N_f)$

 Table 2. Summary of values for the  $\kappa$ -parameter

$N_f = 4$		$N_f = 3$	
$\kappa'$	$\kappa'_m$	$\kappa''$	$\kappa''_m$
$\frac{199}{625} - \frac{28}{225}\pi^2 (\cong -0.909)$	$\frac{7606}{1875} (\cong 4.06)$	$\frac{31}{81} - \frac{28}{243}\pi^2 (\cong -0.753)$	$\frac{290}{81} (\cong 3.58)$

 Table 3. Improved branching ratios for  $\bar{B} \rightarrow K^*\gamma$ 

Model	ref.	$R_0(\%)$	$R'(\%)$	$Br(B \rightarrow K^*\gamma) \times 10^5$
CQM(r)	[6]	4.5	7.4	3.5
CQM & MIT Bag	[8]	6.0	9.8	4.6
CMQ(nr)	[9]	21.	34.	16.1
HQL+CQM	[10]	$3.5 \sim 12.2$	$5.7 \sim 20.0$	$2.7 \sim 9.4$
HQL	[11]	$12. \sim 27.$	$20. \sim 44.$	$9.4 \sim 20.5$